

УДК 517.55

On the Extension of Analytic Sets into a Neighborhood of the Edge of a Wedge in Nongeneral Position

Evgeniya V. Yurieva*

Institute of Mathematics and Computer Science,
Siberian Federal University,
Svobodny, 79, Krasnoyarsk, 660041
Russia

Received 10.05.2013, received in revised form 10.06.2013, accepted 20.06.2013

Let $K = D_+ \cup T^n \cup D_-$ be an n -circled two-sided wedge in \mathbb{C}^n where the unit polycircle (torus) T^n plays a role of the edge, and domains D_\pm adjoined to T^n may not contain any full-dimensional cone near T^n . In this case we say that the wedge K is in nongeneral position. Consider a question when the closures of pure n -dimensional analytic sets $A_\pm \subset D_\pm \times \mathbb{C}^m$ compose a single analytic set in a neighborhood of the wedge $K \times \mathbb{C}^m$. If K is in general position then the answer to the question is given by the theorem of S.I. Pinchuk. In the present article we expand this theorem to the case when the two-circled wedge K is in nongeneral position, and $m = 1$.

Keywords: the edge of the wedge theorem, analytic sets, currents.

Introduction

Besides the Hartogs' theorem, one of the important phenomenon of an analytic continuation of holomorphic functions in several variables is the edge of the wedge theorem obtained by N.N. Bogolubov in 1956 [1, 2]. This theorem asserts that if $f(z)$ is a function holomorphic in a tubular domain $\mathcal{T} = \mathbb{R}^n + i\Gamma$, whose base Γ is the two-sided light cone $y_1^2 > y_2^2 + \dots + y_n^2$, and continuous in its closure $\overline{\mathcal{T}}$, then $f(z)$ admits an analytic continuation in \mathbb{C}^n . In [3] S.I. Pinchuk generalized the Bogolubov's result taking instead of the light cone an arbitrary wedge with the edge on a generating manifold bounded by smooth hypersurfaces in general position. The condition on the general position means that both sides of the wedge contain full-dimensional cones near the points of the edge. In article [4] we studied the problem of holomorphic extension of functions into a neighborhood of the edge of a two-sided n -circled wedge in nongeneral position. It dealt with wedges formed by two n -circled domains D_\pm adjoined to the unit torus $T^n = \{|z_1| = \dots = |z_n| = 1\}$ while the union $K = D_+ \cup T^n \cup D_-$ may not contain any full-dimensional cone near the edge T^n ; in this case we say that the wedge K is in nongeneral position.

The problem of description the removable singularities of analytic sets was considered in works of H. Alexander, J. Bekker, B. Shiffman, K. Funahashi (one can find the statements, proofs and references in the book of E. M. Chirka [5]). Following E. M. Chirka [5] we can pose this problem in a such way: let E be a closed subset of complex manifold X and A is a pure dimensional analytic subset in $X \setminus E$; then a question arises: under what conditions on E and A the closure \overline{A} will be an analytic set in X . The most general result was obtained by E. M. Chirka ([5], Theorem 18.5). He remarked that if E has a rich complex structure then an analytic subset in $X \setminus E$ may approach E with uncontrolled "wobbles". Thus to remove such singularities we need some additional conditions. On the other hand, if the complex structure of E is poor, for example, if E is a smooth manifold that does not contain any maximal complex

*evg_yurieva@fromru.com

© Siberian Federal University. All rights reserved

submanifold of the appropriate dimension (the candidate on the boundary of the analytic set) then one can hope that an analytic subset A in $X \setminus E$ "won't notice" E , i. e. \bar{A} will be analytic in X . This was confirmed by the Shiffman's result [6] about the extension of analytic sets of dimension $p \geq 2$ through the skeleton of the polycircle. K. Funahashi showed [7] that in \mathbb{C}^n the singularities of the type $\mathbb{R}^{n-m} \times \mathbb{C}^m$ are also removable. The Chirka's theorem [5] states that if E is a C^1 -submanifold in X such that the dimension of the complex tangent plane $T_\zeta^c E$ in almost every point ζ is smaller than $p - 1$, and A is a pure p -dimensional analytic subset in $X \setminus E$, then \bar{A} is an analytic subset in X .

S.I. Pinchuk interpreted in [8] the result [3] (on the extension of holomorphic functions) as an phenomenon of continuation of analytic sets defined as graphs of holomorphic functions in a wedge in \mathbb{C}^{n+1} . He received in [8] the generalization of the Bogolubov's theorem for analytic sets defined in a wedge of general position in \mathbb{C}^{n+m} . The goal of this article is to expand the Pinchuk's result to the case of an n -circled wedge in nongeneral position when $n = 2, m = 1$.

1. Admissible analytic subsets in a wedge and their values on the edge

Let us introduce necessary notations. Let Ω be a domain in $\mathbb{C}^3 = \mathbb{C}_z^2 \times \mathbb{C}_w$ of the type

$$\Omega = U(K) \times \omega,$$

where $U(K)$ is a neighborhood of the two-circled wedge of type $K = D_+ \cup T^2 \cup D_- \subset \mathbb{C}_z^2$ and ω is a bounded domain in \mathbb{C}_w . We assume that K contains the "diagonal" $\Delta = \{|z_1| = |z_2|\}$. Let A_\pm be pure two-dimensional analytic subsets in $D_\pm \times \omega$. We will call such subsets *admissible* if the following properties hold:

1. The closures of \bar{A}_\pm do not cross the piece $K \times \partial\omega$ of the boundary $\partial(K \times \omega)$.
2. $A_\pm \cap (\Delta \times \omega)$ has a finite Hausdorff's 3-measure.
3. For any form $\varphi \in \mathcal{D}^{2,0}(\Omega)$ there exist limits

$$\lim_{\varepsilon \rightarrow 0} \int_{(T_{1 \pm \varepsilon}^2 \times \omega) \cap A_\pm} \varphi$$

defining currents $\partial^0 A_\pm$ of bidimension $(2, 0)$ on Ω .

Here $T_{1 \pm \varepsilon}^2 = \{|z_1| = |z_2| = 1 \pm \varepsilon\}$ is a family of tori which lie on the "diagonal" Δ and exhaust it. These currents we will call *the values* of A_\pm on $T^2 \times \omega$. We say that values of A_+ and A_- coincide on $T^2 \times \omega$, if $\partial^0 A_+ = \partial^0 A_-$ (see Fig. 1).

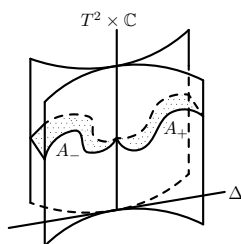


Fig. 1

Remark that the given notion of admissible subsets is inspired by the definition of S. I. Pinchuk [8], but they are different. The essential point is that we are able to introduce the currents $\partial^0 A_\pm$ without any restriction condition on the wedge K , like a restriction "of a general position".

Theorem. Assume that the wedge $K = D_+ \cup T^2 \cup D_-$ contains the "diagonal" Δ and A_\pm are admissible pure two-dimensional analytic sets in $D_\pm \times \omega$ with the same values on $T^2 \times \omega$:

$$\partial^0 A_+ = \partial^0 A_-.$$

Then the closures \bar{A}_+ and \bar{A}_- compose a single analytic set in a neighborhood of $K \times \omega$.

2. The proof of the theorem

In view of the property 1 for admissible analytic sets, the projection $\pi : A_\pm \rightarrow D_\pm$ is a proper mapping and therefore it is a branched covering. Hence the sets A_\pm are defined by Weierstrass' polynomials

$$\prod_{\nu} \langle w - w_\nu(z) \rangle = w^k + a_1^+(z)w^{k-1} + \cdots + a_k^+(z) = 0,$$

$$\prod_{\mu} \langle w - w_\mu(z) \rangle = w^l + a_1^-(z)w^{l-1} + \cdots + a_l^-(z) = 0$$

with $a_j^\pm(z)$ being holomorphic functions in D_\pm .

Denote $\rho_1(z) := |z_1| - |z_2|$ and let $\rho_2(z) = \rho_2(|z_1|, |z_2|)$ be a smooth function in $U(K)$ such that the image of $\{\rho_2 = 0\}$ on the Rheinhard diagram lies in D_+ below the diagonal Δ and in D_- above Δ (see Fig. 2). Using functions ρ_1 and ρ_2 we introduce the following domains:

$$\Omega_{\pm i} = \{(z, w) \in \Omega : \pm \rho_i(z) > 0\}, \quad i = 1, 2, \quad \Omega_\pm = \Omega_{\pm 1} \cap \Omega_{\pm 2}.$$

In domains $\Omega_{\pm i}$ we define currents $T_{\pm i}$ of bidimension $(2, 2)$:

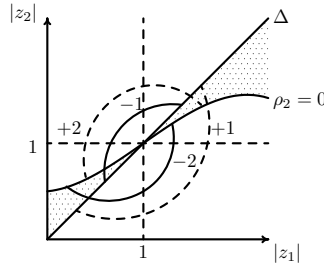


Fig. 2

$$\langle T_{\pm 2}, \varphi \rangle := \pm \int_{\Omega_\pm \cap A_\pm} \varphi, \quad (\varphi \in \mathcal{D}^{2,2}(\Omega_{\pm 2})), \quad T_{\pm 1} := 0, \quad (1)$$

and currents A_{Δ_\pm} of bidimension $(2, 1)$:

$$\langle A_{\Delta_\pm}, \varphi \rangle := \pm \int_{(\Delta_\pm \times \omega) \cap A_\pm} \varphi, \quad (2)$$

here $\Delta_+ = \{|z_1| = |z_2| > 1\}$ and $\Delta_- = \{|z_1| = |z_2| < 1\}$. By property 2 the last integrals are well defined.

Since A_{\pm} are branched coverings over D_{\pm} , the points $(z, w) \in A_{\pm}$ are parametrized by $z \in D_{\pm}$, therefore we can consider $A_{\Delta_{\pm}}$ as currents on Δ defined on forms in $\mathcal{D}^{2,1}(U(K))$. Similarly we consider $T_{\pm i}$ as currents on $\Omega'_{\pm i}$ defined on forms in $\mathcal{D}^{2,2}(\Omega'_{\pm i})$, where $\Omega'_{\pm i}$ are the images of $\Omega_{\pm i}$, i. e.

$$\Omega'_{\pm i} = \{z \in U(K) : \pm \rho_i(z) > 0\}, \quad i = 1, 2, \quad \Omega'_{\pm} = \Omega'_{\pm 1} \cap \Omega'_{\pm 2}.$$

Taking into account this remark we rewrite the sum of currents in (2) as

$$\langle A_{\Delta}, \varphi \rangle := \lim_{\varepsilon \rightarrow 0} \left(\int_{\Delta_{-}^{\varepsilon}} \varphi + \int_{\Delta_{+}^{\varepsilon}} \varphi \right),$$

where $\Delta_{-}^{\varepsilon} = \{|z_1| = |z_2| \leq 1 - \varepsilon\}$, $\Delta_{+}^{\varepsilon} = \{|z_1| = |z_2| \geq 1 + \varepsilon\}$. By property 3 and by assumption of Theorem we have

$$\langle A_{\Delta}, \varphi \rangle := \lim_{\varepsilon \rightarrow 0} \left(\int_{\Delta_{-}^{\varepsilon}} \varphi + \int_{\Delta_{+}^{\varepsilon}} \varphi \right),$$

i. e. A_{Δ} is a $\bar{\partial}$ -closed current.

For $q = 1, 2, \dots$ we define the current $\sigma^q = w^q A_{\Delta}$. Of course, σ^q is $\bar{\partial}$ -closed. In a small Stein neighborhood $U(\Delta)$ it is $\bar{\partial}$ -exact:

$$\sigma^q = \bar{\partial} \tau^q, \text{ where } \tau^q \in \mathcal{D}'(U(K)).$$

Here we used the distribution space notation \mathcal{D}' since currents of bidimension $(2, 2)$ in $U(K)$ are generalized functions.

Let $\tau_{\pm i}^q := w^q T_{\pm i} - \tau^q$. We will show that these currents are $\bar{\partial}$ -closed in $\Omega'_{\pm i}$. For $\varphi \in \mathcal{D}^{2,1}(\Omega'_{\pm 2})$ one has

$$\langle \bar{\partial} \tau_{\pm 2}^q, \varphi \rangle = \langle \bar{\partial} w^q T_{\pm 2} - \bar{\partial} \tau^q, \varphi \rangle = \int_{\Omega'_{\pm}} w^q \bar{\partial} \varphi - \langle \sigma^q, \varphi \rangle = \int_{\Omega'_{\pm}} d(w^q \varphi) - \langle \sigma^q, \varphi \rangle.$$

Using the Stokes' formula we get

$$\langle \bar{\partial} \tau_{\pm 2}^q, \varphi \rangle = \int_{\Delta} w^q \varphi - \int_{\rho_2=0} w^q \varphi - \langle \sigma^q, \varphi \rangle,$$

where the first integral gives $\langle \sigma^q, \varphi \rangle$ and the second one vanishes, because having a compact support in $\Omega'_{\pm 2}$ the differential form $\varphi = 0$ in a neighborhood of $\{\rho_2 = 0\}$. Hence, $\bar{\partial} \tau_{\pm 2}^q = 0$ in $\Omega'_{\pm 2}$. Similarly, $\bar{\partial} \tau_{\pm 1}^q = 0$ in $\Omega'_{\pm 1}$. So, $\tau_{\pm i}^q$ are holomorphic functions in $\Omega'_{\pm i}$.

Currents $\tau_{\pm}^q = \tau_{\pm 2}^q - \tau_{\pm 1}^q$ are equal to $w^q T_{\pm 2}$, i. e. they act in Ω'_{\pm} by formulas

$$\langle \tau_{\pm}^q, \varphi \rangle = \int_{\Omega'_{\pm}} \left[\sum_{\nu} (w_{\nu}(z))^q \right] \varphi, \quad \varphi \in \mathcal{D}^{2,2}(\Omega'_{\pm}),$$

where w_{ν} are the roots of the corresponding Weierstrass' polynomial, so τ_{\pm}^q are polynomials in coefficients $a_{\mu}^{\pm}(z)$.

Note that τ_{+2}^q and τ_{-1}^q, τ_{+1}^q and τ_{-2}^q coincide and are equal to $-\tau^q$ in the intersections $\Omega'_{+2} \cap \Omega'_{-1}, \Omega'_{+1} \cap \Omega'_{-2}$ correspondingly. So they form holomorphic functions in $\Omega'_{+2} \cup \Omega'_{-1}, \Omega'_{+1} \cup \Omega'_{-2}$ correspondingly. By Bochner's theorem [1] these functions admit an analytic continuation into the convex hull of the wedge K . So we get that coefficients of Weierstrass' polynomials are the same: $a_{\mu}^{+}(z) = a_{\mu}^{-}(z)$. Thereby we constructed a Weierstrass' polynomial that defines a single analytic set in a neighborhood of $K \times \omega$.

References

- [1] V.S.Vladimirov, Methods of the Theory of Functions of Complex Variables, Nauka, Moscow, 1964 (in Russian).
- [2] N.N.Bogolubov, B.V.Medvedev, M.K.Polivanov, Problems of the Theory of Dispersion Relations, Fizmatgiz, Moscow, 1958 (in Russian).
- [3] S.I.Pinchuk, Bogolubov's theorem on the "edge of the wedge" for generic manifolds, *Mat. Sb.*, **94(136)**(1974), no. 3(7), 468–482 (in Russian).
- [4] E.V.Yurieva, On the holomorphic extension into a neighborhood of the edge of a wedge in nongeneral position, *Siberian Math. J.*, **52**(2011), no. 3, 563–568.
- [5] E.M.Chirka, Complex Analytic Sets, Nauka, Moscow, 1985 (in Russian).
- [6] B.Shiffman, On the continuation of analytic sets, *Math. Ann.*, **185**(1970), 1–12.
- [7] K.Funahashi, On the extension of analytic sets, *Proc. Japan Acad.*, **54**(1978), Ser. A., 24–26.
- [8] S.I.Pinchuk, The edge of the wedge theorem for analytic sets, *Dokl. Akad. Nauk SSSR*, **285**(1985), 563–566 (in Russian).

Продолжение аналитических множеств в окрестность острого клина необщего положения

Евгения В. Юрьева

Пусть $K = D_+ \cup T^n \cup D_-$ n -круговой двусторонний клин в \mathbb{C}^n с острием на осто́ве T^n единичного поликруга. При этом примыкающие к остову области D_{\pm} могут не содержать вблизи T^n никакого полномерного конуса. В этом случае мы говорим, что K — клин необщего положения. Рассматривается вопрос о том, когда чисто n -мерные аналитические множества $A_{\pm} \subset D_{\pm} \times \mathbb{C}^m$ продолжаются до единого аналитического множества в окрестности клина $K \times \mathbb{C}^m$. Если K — клин общего положения, то ответ на поставленный вопрос дает теорема С. И. Пинчука. В статье для случая $n = 2, m = 1$ эта теорема распространяется на клин необщего положения.

Ключевые слова: теорема об остром клине, аналитические множества, потоки.